Cotorsion Modules and a Problem of George Bergman

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Introduction

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- **Problem** (Stated for modules): Let R be a ring with identity. Let $P = \prod_{i \in I} A_i$ where $A_i = R$ for all i and let $S = \bigoplus_{i \in I} A_i$, considered as left R-modules. Describe the left R-modules M which have the property that every homomorphism $f : S \longrightarrow M$ extends to a homomorphism $g : P \longrightarrow M$.

Cotorsion abelian groups

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- **Examples:** (i) Any finite abelian group; (ii) $\oplus\mathbb{Q}$ or any injective $\mathbb{Z}$-module; (iii) $\prod_p \mathbb{Z}(p)$; $\prod_{n\in\mathbb{N}} F_n$, F_n a finite group; (iv) Any abelian group admitting a compact group topology; (v) Homomorphic images of pure-injective abelian groups; (vi) $\mathbb{Z}$ is **NOT** a cotorsion group.

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- He showed that there is a categorical equivalence between "reduced" torsion groups and "adjusted" cotorsion groups by $T \longmapsto Ext(\mathbb{Q}/\mathbb{Z}, T)$ and $C \longmapsto Tor(\mathbb{Q}/\mathbb{Z}, C)$. The functors *Ext* and *Tor* act as inverse functors on these categories:

$$Tor(\mathbb{Q}/\mathbb{Z}, Ext(\mathbb{Q}/\mathbb{Z}, T)) \cong T \text{ and } Ext(\mathbb{Q}/\mathbb{Z}, Tor(\mathbb{Q}/\mathbb{Z}, C) \cong C.$$

Properties of P

- Let $P = \prod_{n \in \mathbb{N}} \mathbb{Z} = \prod_{n \in \mathbb{N}} < e_n >$ be a direct product of infinite cyclic (abelian) groups $< e_n >$, where $\mathbb{N}$ denotes the set $\{1, 2, 3, \cdots\}$.

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- REFERENCE: L. Fuchs, Abelian Groups, Springer Monographs in Math, Springer (2015).

The Extension Property

- **Theorem 1**: Let G be an abelian group, $P = \prod_{n \in \mathbb{N}} < e_n >$ and $S = \bigoplus_{n \in \mathbb{N}} < e_n >$ where $< e_n > \cong \mathbb{Z}$ for all n. Suppose every homomorphism $f : S \longrightarrow G$ extends to a homomorphism $g : P \longrightarrow G$. Then every countable subgroup of G embeds in a cotorsion subgroup of G.

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- Let $\mathbb{N} = \bigcup_{k \geq 1} X_k$ be an infinite partition of $\mathbb{N}$ where $|X_k| = \aleph_0$ for all k. Then we can write $S = \bigoplus_{k \geq 1} S_k$ where $S_k = \bigoplus_{n \in X_k} < e_n >$.

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- Define a homomorphism $f : S \longrightarrow G$ by $f(e_n) = a_k$ for all $n \in X_k$, so $f(S_k) =< a_k >$.

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- Define a homomorphism $f : S \longrightarrow G$ by $f(e_n) = a_k$ for all $n \in X_k$, so $f(S_k) =< a_k >$.
- **Claim**: This f does not extend to a homomorphism $g : P \longrightarrow G$.

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- **Claim**: This f does not extend to a homomorphism $g : P \longrightarrow G$.
- Suppose such a g exists. Now, $A \subseteq im(g) = B \oplus C$, where $B \cong P$ or is finitely generated free and C is cotorsion (Property 3). By supposition, $A \nsubseteq C$ and so if $\pi : B \oplus C \longrightarrow B$ is the coordinate projection with $\ker(\pi) = C$, then $\pi(a_k) \neq 0$ for some k. By Property 2., $\pi(a_k) \in D$, a finitely generated free summand of B. If $\pi' : B \longrightarrow D$ is a coordinate projection, then we have a homomorphism $\pi'\pi g : P \xrightarrow{g} B \oplus C \xrightarrow{\pi} B \xrightarrow{\pi'} D$. By Property 1., $\pi'\pi g(e_n) = 0$ for all except finitely many $n \in \mathbb{N}$. This is a contradiction since for infinitely many $e_n \in X_k$, $\pi'\pi g(e_n) = \pi'\pi f(e_n) = \pi'\pi(a_k) \neq 0$. So $A \subseteq C$.

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- Thus for abelian groups, we have a solution to Bergman's Problem.
- **Corollary 3**: Let G be an abelian group. Every homomorphism $f : S \to G$ extends to a homomorphism $g : P \to G$ if and only if G is a cotorsion group.

Generalization to Domains

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- Our goal is to show that, under certain conditions, each of these classes of R-modules satisfy the Bergman extension property.
- First, some Homological Preliminaries:

- A sequence of modules and maps of the form $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be **exact** at B if $im(f) = \ker(g)$. Thus $0 \rightarrow A \xrightarrow{f} B$ exact at A means that $\ker(f) = \{0\}$. Likewise, $B \xrightarrow{g} C \rightarrow 0$ is exact at C means that $im(g) = \ker(C \rightarrow 0) = C$. Thus if A is a submodule of B and $B/A = C$ we have an exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\eta} C \rightarrow 0$ where i is the inclusion map and η is the natural coset map.

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- If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is an exact sequence, we call B **an extension of** A **by** C. A second extension $0 \rightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \rightarrow 0$ is said to be **equivalent** to the preceding one if there is an isomorphism $\phi : B \rightarrow B'$ such that the following diagram is commutative

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- Given A and C, the set of inequivalent extensions of A by C form an abelian group denoted by $Ext^1_R(C, A)$ whose zero element is $A \oplus C$.

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- $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] & C \arrow[r] & 0 \\ & A' & & & \end{tikzcd}$$

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- Similarly, using a **Pullback** diagram, a homomorphism $g : C' \rightarrow C$ induces a homomorphism $Ext^1_R(C, A) \xrightarrow{\bar{g}} Ext^1_R(C', A)$. For a fixed A, $Ext^1_R(-, A)$ is a (contravariant) functor from the category **M** to the category **A** of abelian groups.

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- Let C_κ be the closure of S_κ in the product topology of P_κ. It can be shown that $C_\kappa / S_\kappa \cong \bigoplus_{\kappa^{\aleph_0}} Q = \bigoplus_{2^\kappa} Q$

- **Proposition 5**: Let C be an R-module of cardinality $\leq 2^{\kappa}$. Then C is a Matlis cotorsion module, that is, $Ext_R^1(Q, C) = 0$ if and only if $Ext_R^1(C_\kappa, C) = 0$.

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- Assume $Ext_R^1(C_\kappa, C) = 0$. Then we get the exact sequence $Hom_R(S_\kappa, C) \to Ext_R^1(C_\kappa/S_\kappa, C) \to 0$. So $|Hom_R(S_\kappa, C)| \geq |Ext_R^1(C_\kappa/S_\kappa, C)|$. Suppose, by way of contradiction, $Ext_R^1(Q, C) \neq 0$. Now $|Hom_R(S_\kappa, C)| = |Hom_R(\bigoplus_\kappa R, C)| = |\prod_\kappa Hom_R(R, C)| = |\prod_\kappa C| \leq (2^\kappa)^\kappa = 2^\kappa$. On the other hand, since $Ext_R^1(Q, C) \neq 0$, $|Ext_R^1(C_\kappa/S_\kappa, C)| = |Ext_R^1(\bigoplus_{2^\kappa} Q, C)| = \prod_{2^\kappa} |Ext_R^1(Q, C)| \geq 2^{2^\kappa}$. We get a contradiction, since $2^\kappa \ngeq 2^{2^\kappa}$. Thus $Ext_R^1(Q, C) = 0$.

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